$$\cos(\alpha + \beta) \equiv \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

if $\alpha, \beta \in R$. To do this we use the unit circle and the distance formula. Consider Figure 2.1, where the arcs $\widehat{PQ}$ and $\widehat{PR}$ are assumed to have lengths α and β, respectively. This means that the points Q and R must have the coordinates indicated. The points S and T depend upon the points P and Q. S is the point

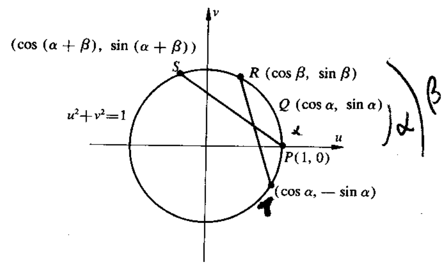

Figure 2.1

corresponding to $\alpha + \beta$, and T is the point corresponding to $-\alpha$. This determines the coordinates for the points S and T. Note that identities for $\cos(-\alpha)$ and $\sin(-\alpha)$ have been used in giving the coordinates of T.

Because of the way in which the points S and T were selected, the arcs $\widehat{PS}$ and $\widehat{TR}$ have the same length, $\alpha + \beta$. Since, within the same circle, equal arcs subtend equal chords, the lengths of the segments $\overline{PS}$ and $\overline{TR}$ are equal. This leads us to the following development.

$$\begin{aligned} d(PS) &= \sqrt{[\cos(\alpha+\beta) - 1]^2 + [\sin(\alpha+\beta) - 0]^2} \\ &= \sqrt{\cos^2(\alpha+\beta) - 2\cos(\alpha+\beta) + 1 + \sin^2(\alpha+\beta)} \\ &= \sqrt{2 - 2\cos(\alpha+\beta)}. \\ d(TR) &= \sqrt{(\cos\beta - \cos\alpha)^2 + [\sin\beta - (-\sin\alpha)]^2} \\ &= \sqrt{(\cos\beta - \cos\alpha)^2 + (\sin\beta + \sin\alpha)^2} \\ &= \sqrt{\cos^2\beta - 2\cos\beta\cos\alpha + \cos^2\alpha + \sin^2\beta + 2\sin\beta\sin\alpha + \sin^2\alpha} \\ &= \sqrt{2 - 2\cos\beta\cos\alpha + 2\sin\beta\sin\alpha}. \end{aligned}$$

Since $d(PS) = d(TR)$, we have

$$\begin{aligned} \sqrt{2 - 2\cos(\alpha+\beta)} &= \sqrt{2 - 2\cos\beta\cos\alpha + 2\sin\beta\sin\alpha}, \\ 2 - 2\cos(\alpha+\beta) &= 2 - 2\cos\beta\cos\alpha + 2\sin\beta\sin\alpha, \\ -2\cos(\alpha+\beta) &= -2\cos\beta\cos\alpha + 2\sin\beta\sin\alpha, \\ \cos(\alpha+\beta) &= \cos\beta\cos\alpha - \sin\beta\sin\alpha, \\ \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta. \end{aligned}$$

$$\therefore \cos(\alpha+\beta) \equiv \cos\alpha\cos\beta - \sin\alpha\sin\beta, \quad \alpha \in R, \beta \in R.$$

Any real numbers α and β could have been selected. α and β were chosen to be positive with a sum less than π for the convenience of the illustration. You may wish to experiment with other real numbers to demonstrate the generality of the identity for yourself.

If we wish to use an expression for $\cos(\alpha - \beta)$, we note that $\alpha - \beta = \alpha + (-\beta)$. This enables us to develop

$$\cos(\alpha - \beta) \equiv \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

as follows:

$$\begin{aligned} \cos(\alpha - \beta) &= \cos[\alpha + (-\beta)] \\ &= \cos\alpha\cos(-\beta) - \sin\alpha\sin(-\beta) \\ &= \cos\alpha\cos\beta - \sin\alpha(-\sin\beta) \\ &= \cos\alpha\cos\beta + \sin\alpha\sin\beta. \end{aligned}$$

$$\therefore \cos(\alpha - \beta) \equiv \cos\alpha\cos\beta + \sin\alpha\sin\beta, \quad \alpha \in R, \beta \in R.$$

From the identity $\cos(\alpha - \beta) \equiv \cos\alpha\cos\beta + \sin\alpha\sin\beta$ we can develop the identity

$$\cos\left(\frac{\pi}{2} - \theta\right) \equiv \sin\theta.$$

This identity will be most helpful in developing expressions for $\sin(\alpha + \beta)$ and $\sin(\alpha - \beta)$. We note that

$$\cos\left(\frac{\pi}{2} - \theta\right) = \cos\frac{\pi}{2}\cos\theta + \sin\frac{\pi}{2}\sin\theta$$
$$= (0)\cos\theta + (1)\sin\theta$$
$$= \sin\theta.$$

$$\therefore \cos\left(\frac{\pi}{2} - \theta\right) \equiv \sin\theta, \quad \mathscr{D} = R.$$

It is also true that $\sin(\pi/2 - \theta) \equiv \cos\theta$ for any real number θ. We note that

$$\cos\left[\frac{\pi}{2} - \left(\frac{\pi}{2} - \theta\right)\right] = \sin\left(\frac{\pi}{2} - \theta\right).$$

But since

$$\frac{\pi}{2} - \left(\frac{\pi}{2} - \theta\right) = \frac{\pi}{2} - \frac{\pi}{2} + \theta = \theta,$$

we have

$$\cos\theta = \cos\left[\frac{\pi}{2} - \left(\frac{\pi}{2} - \theta\right)\right]$$
$$= \sin\left(\frac{\pi}{2} - \theta\right).$$

$$\therefore \sin\left(\frac{\pi}{2} - \theta\right) \equiv \cos\theta, \quad \mathscr{D} = R.$$

We now note that $\cos[\pi/2 - (\alpha + \beta)] = \sin(\alpha + \beta)$. But since

$$\frac{\pi}{2} - (\alpha + \beta) = \left(\frac{\pi}{2} - \alpha\right) - \beta,$$

we have

$$\sin(\alpha + \beta) = \cos\left[\frac{\pi}{2} - (\alpha + \beta)\right]$$
$$= \cos\left[\left(\frac{\pi}{2} - \alpha\right) - \beta\right]$$
$$= \cos\left(\frac{\pi}{2} - \alpha\right)\cos\beta + \sin\left(\frac{\pi}{2} - \alpha\right)\sin\beta$$
$$= \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

$$\therefore \sin(\alpha + \beta) \equiv \sin\alpha\cos\beta + \cos\alpha\sin\beta, \quad \alpha \in R, \beta \in R.$$

To develop an expression for $\sin(\alpha - \beta)$, we note that $\alpha - \beta = \alpha + (-\beta)$ and use identities involving $\sin(-\beta)$ and $\cos(-\beta)$. The development of

$$\sin(\alpha - \beta) \equiv \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

appears below.